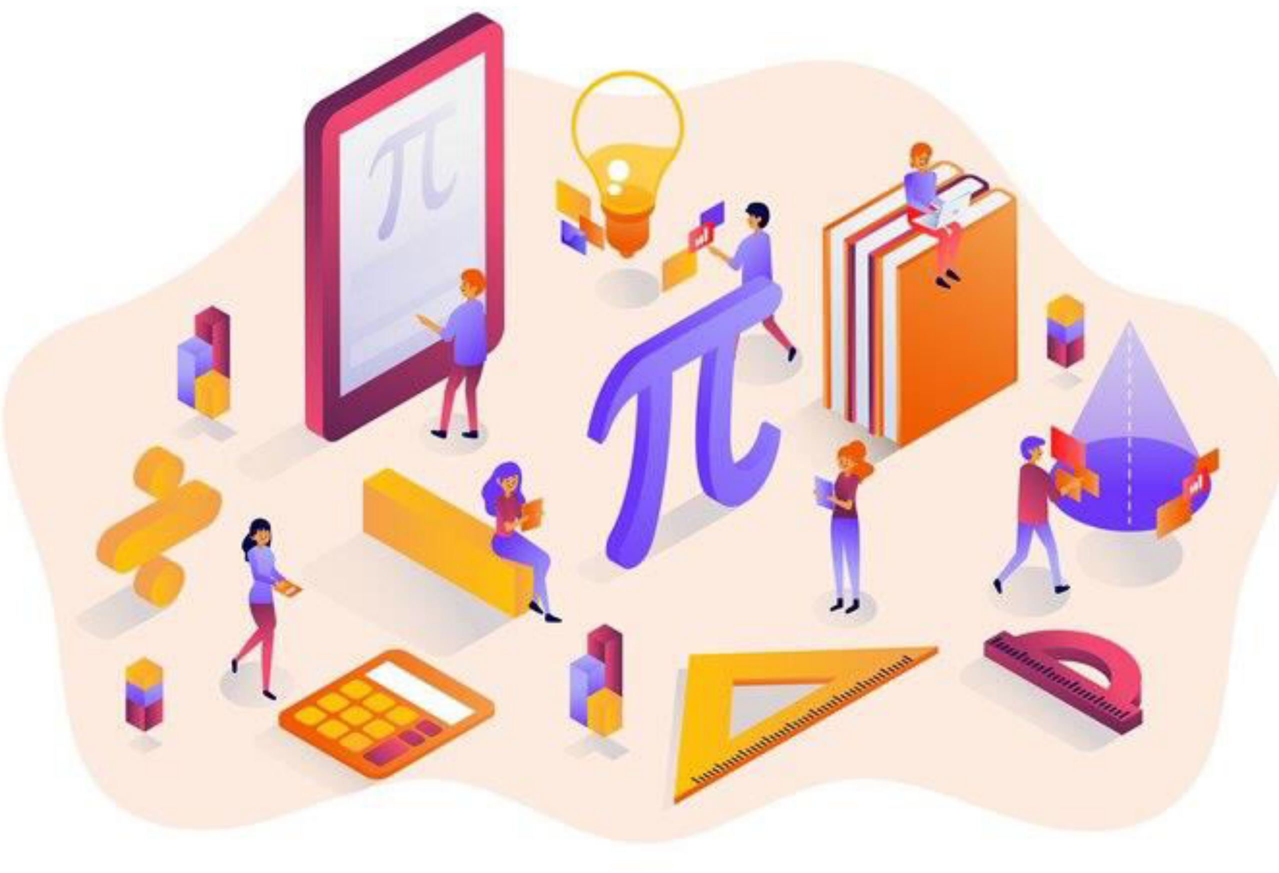
MATHEMATICS



Continuity & Differentiability

Top Definitions

1. A function f(x) is said to be continuous at a point c if
$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

2. A real function f is said to be continuous if it is continuous at every point in the domain of f.

3. If f and g are real-valued functions such that (f o g) is defined at c, then
$(f \circ g)(x) = f(g(x))$.
If g is continuous at c and if f is continuous at g(c), then (f o g) is continuous at c.

4. A function f is differentiable at a point c if Left Hand Derivative (LHD) = Right Hand Derivative (RHD),
i.e. $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$

5. If a function f is differentiable at every point in its domain, then
$\lim_{h \to 0} \frac{f(x+h) - f(c)}{h}$ or $\lim_{h \to 0} \frac{f(x-h) - f(c)}{-h}$ is called the derivative or differentiation of f at x and is denoted by $f'(x)$ or $\frac{d}{dx} f(x)$.

6. If $LHD \neq RHD$, then the function f(x) is not differentiable at x = c.

7. Geometrical meaning of differentiability:
The function f(x) is differentiable at a point P if there exists a unique tangent at point P. In other words, f(x) is differentiable at a point P if the curve does not have P as its corner point.

8. A function is said to be differentiable in an interval (a, b) if it is differentiable at every point of (a, b).

9. A function is said to be differentiable in an interval [a, b] if it is differentiable at every point of [a, b].

10. Chain Rule of Differentiation: If f is a composite function of two functions u and v such that f = v(t) and t = u(x) and if both $\frac{dv}{dt}$ and $\frac{dt}{dx}$ exist, then $\frac{dv}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$.

11. Logarithm of a to the base b is x, i.e. $\log_b a = x$ if $b^x = a$, where b > 1 is a real number. Logarithm of a to base b is denoted by $\log_b a$.

12. Functions of the form x = f(t) and y = g(t) are parametric functions.

13. **Rolle's Theorem**: If f : [a, b] → **R** is continuous on [a, b] and differentiable on (a, b) such that f(a) = f(b), then there exists some c in (a, b) such that f'(c) = 0.

14. **Mean Value Theorem**: If f : [a, b] → **R** is continuous on [a, b] and differentiable on (a, b), then there exists some c in (a, b) such that $f^{'}(c) = \lim_{h \to 0} \frac{f(b) - f(a)}{b - a}$.

Top Concepts

1. A function is continuous at x = c if the function is defined at x = c and the value of the function at x = c equals the limit of the function at x = c.

2. If function f is not continuous at c, then f is discontinuous at c and c is called the point of discontinuity of f.

3. Every polynomial function is continuous.

4. The greatest integer function [x] is not continuous at the integral values of x.

5. Every rational function is continuous.

   Algebra of continuous functions
   1. Let f and g be two real functions continuous at a real number c, then
      f + g is continuous at x = c.
   2. f – g is continuous at x = c.
   3. f . g is continuous at x = c.
   4. $\left(\frac{f}{g}\right)$ is continuous at x = c, [provided g(c) ≠ 0].
   5. kf is continuous at x = c, where k is a constant.

6. Consider the following functions:
   1. Constant function
   2. Identity function
   3. Polynomial function
   4. Modulus function
   5. Exponential function
   6. Sine and cosine functions

   The above functions are continuous everywhere.

7. Consider the following functions:
   1. Logarithmic function
   2. Rational function
   3. Tangent, cotangent, secant and cosecant functions

   The above functions are continuous in their domains.

8. If f is a continuous function, then $|f|$ and $\frac{1}{f}$ are continuous in their domains.

9. Inverse functions $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x, \cot^{-1}x, \operatorname{cosec}^{-1}x$ and $\sec^{-1}x$ are continuous functions on their respective domains.

10. The derivative of a function f with respect to x is f′(x) which is given by $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$.

11. If a function f is differentiable at a point c, then it is also continuous at that point.

12. Every differentiable function is continuous, but the converse is not true.

13. Every polynomial function is differentiable at each $x \in R$.

14. Every constant function is differentiable at each $x \in R$.

15. The chain rule is used to differentiate composites of functions.

16. The derivative of an even function is an odd function and that of an odd function is an even function.

17. **Algebra of Derivatives**

If u and v are two functions which are differentiable, then

i. $(u \pm v)' = u' \pm v'$ (Sum and Difference Formula)

ii. $(uv)' = u'v + uv'$ (Leibnitz rule or Product rule)

iii. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, $v \neq 0$, (Quotient rule)

18. **Implicit Functions**

If it is not possible to separate the variables x and y, then the function f is known as an implicit function.

19. **Exponential function:** A function of the form $y = f(x) = b^x$, where base b > 1.

(1) Domain of the exponential function is R, the set of all real numbers.

(2) The point (0, 1) is always on the graph of the exponential function.

(3) The exponential function is ever increasing.

20. The exponential function is differentiable at each $x \in R$.

21. Properties of logarithmic functions:

i. Domain of log function is R^+.

ii. The log function is ever increasing.

iii. For ‘x’ very near to zero, the value of log x can be made lesser than any given real number.

22. Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$. Here both f(x) and u(x) need to be positive.

23. To find the derivative of a product of a number of functions or a quotient of a number of functions, take the logarithm of both sides first and then differentiate.

24. **Logarithmic Differentiation**

$y = a^x$

Taking logarithm on both sides

$\log y = \log a^x$.

Using the property of logarithms

$\log y = x \log a$

Now differentiating the implicit function

$\frac{1}{y} . \frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a = a^x \log a$

25. The logarithmic function is differentiable at each point in its domain.

26. Trigonometric and inverse-trigonometric functions are differentiable in their respective domains.

27. The sum, difference, product and quotient of two differentiable functions are differentiable.

28. The composition of a differentiable function is a differentiable function.

29. A relation between variables x and y expressed in the form x = f(t) and y = g(t) is the parametric form with t as the parameter. Parametric equation of parabola $y^2 = 4ax$ is $x = at^2$, $y = 2at$.

30. Differentiation of an infinite series: If f(x) is a function of an infinite series, then to differentiate the function f(x), use the fact that an infinite series remains unaltered even after the deletion of a term.

31. **Parametric Differentiation:**

Differentiation of the functions of the form x = f(t) and y = g(t):

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

32. Let u = f(x) and v = g(x) be two functions of x. Hence, to find the derivative of f(x) with respect g(x), we use the following formula:

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

33. If y = f(x) and $\frac{dy}{dx}$ = f'(x) and if f'(x) is differentiable, then

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ or f''(x) is the second order derivative of y with respect to x.

34. If x = f(t) and y = g(t), then

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{g'(t)}{f'(t)}\right\}$$

or $$\frac{d^2y}{dx^2} = \frac{d}{dt}\left\{\frac{g'(t)}{f'(t)}\right\} \cdot \frac{dt}{dx}$$

or, $$\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{\{f'(t)\}^3}$$

Top Formulae

1. **Derivative of a function at a point**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2. **Properties of logarithms**

$$\log(xy) = \log x + \log y$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\log\left(x^y\right) = y \log x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

3. **Derivatives of Functions**

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}, a > 0, a \neq 1$$

$$\frac{d}{dx}\left(a^x\right)=a^x\log_e a, a>0$$

$$\frac{d}{dx}\left(\sin^{-1}x\right)=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1}x\right)=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\cot^{-1}x\right)=-\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}},\ \text{if } |x|>1$$

$$\frac{d}{dx}\left(\operatorname{co\,sec}^{-1}x\right)=\frac{-1}{x\sqrt{x^2-1}},\ \text{if } |x|>1$$

$$\frac{d}{dx}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right\}=\begin{cases}-\dfrac{2}{1+x^2}, x>1\\ \dfrac{2}{1+x^2}, -1<x<1\\ -\dfrac{2}{1+x^2}, x<-1\end{cases}$$

$$\frac{d}{dx}\left\{\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}=\begin{cases}\dfrac{2}{1+x^2}, x>0\\ \dfrac{-2}{1+x^2}, x<0\end{cases}$$

$$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}=\begin{cases}\dfrac{2}{1+x^2}, x<-1 \text{ or } x>1\\ \dfrac{2}{1+x^2}, -1<x<1\end{cases}$$

$$\frac{d}{dx}\left\{\sin^{-1}\left(3x-4x^3\right)\right\}=\begin{cases}-\dfrac{3}{\sqrt{1-x^2}}, \dfrac{1}{2}<x<1, -1<x<-\dfrac{1}{2}\\ \dfrac{3}{\sqrt{1-x^2}}, -\dfrac{1}{2}<x<\dfrac{1}{2}\end{cases}$$

$$\frac{d}{dx}\left\{\cos^{-1}\left(4x^3-3x\right)\right\}=\begin{cases}-\dfrac{3}{\sqrt{1-x^2}}, \dfrac{1}{2}<x<1\\ \dfrac{3}{\sqrt{1-x^2}}, -\dfrac{1}{2}<x<\dfrac{1}{2} \text{ or } -1<x<-\dfrac{1}{2}\end{cases}$$

$$\frac{d}{dx}\left\{\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)\right\}=\begin{cases}\dfrac{3}{1+x^2}, x<-\dfrac{1}{\sqrt{3}} \text{ or } x>\dfrac{1}{\sqrt{3}}\\ \dfrac{3}{1+x^2}, -\dfrac{1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}\end{cases}$$

$$\frac{d}{dx}\left[\sin\left(\sin^{-1}x\right)\right]=1,\ \text{if}\ -1<x<1$$

$$\frac{d}{dx}\left[\cos\left(\cos^{-1}x\right)\right]=1,\ \text{if}\ -1<x<1$$

$$\frac{d}{dx}\left[\tan\left(\tan^{-1}x\right)\right]=1,\ \text{for all}\ x\in R$$

$$\frac{d}{dx}\left[\operatorname{cosec}\left(\operatorname{cosec}^{-1}x\right)\right]=1,\ \text{for all}\ x\in R-(-1,1)$$

$$\frac{d}{dx}\left[\sec\left(\sec^{-1}x\right)\right]=1,\ \text{for all}\ x\in R-(-1,1)$$

$$\frac{d}{dx}\left[\cot\left(\cot^{-1}x\right)\right]=1,\ \text{for all}\ x\in R$$

$$\frac{d}{dx}\left[\sin^{-1}(\sin x)\right]=\begin{cases}-1, -\frac{3\pi}{2}<x<-\frac{\pi}{2}\\ 1, -\frac{\pi}{2}<x<\frac{\pi}{2}\\ -1, \frac{\pi}{2}<x<\frac{3\pi}{2}\\ 1, \frac{3\pi}{2}<x<\frac{5\pi}{2}\end{cases}$$

$$\frac{d}{dx}\left[\cos^{-1}(\cos x)\right]=\begin{cases}1, 0<x<\pi\\ -1, \pi<x<2\pi\end{cases}$$

$$\frac{d}{dx}\left[\tan^{-1}(\tan x)\right]=\left\{1, n\pi-\frac{\pi}{2}<x<\frac{\pi}{2}+n\pi, n\in Z\right.$$

$$\frac{d}{dx}\left[\operatorname{cosec}^{-1}(\operatorname{cosec}x)\right]=\begin{cases}1, -\frac{\pi}{2}<x<0\ \text{or}\ 0<x<\frac{\pi}{2}\\ -1, \frac{\pi}{2}<x<\pi\ \text{or}\ \pi<x<\frac{3\pi}{2}\end{cases}$$

$$\frac{d}{dx}\left[\sec^{-1}(\sec x)\right]=\begin{cases}1, 0<x<\frac{\pi}{2}\ \text{or}\ \frac{\pi}{2}<x<\pi\\ -1, \pi<x<\frac{3\pi}{2}\ \text{or}\ \frac{3\pi}{2}<x<2\pi\end{cases}$$

$$\frac{d}{dx}\left[\cot^{-1}(\cot x)\right]=1, (n-1)\pi<x<n\pi, n\in Z$$

4. **Differentiation of constant functions**
   1. Differentiation of a constant function is zero, i.e.

      $$\frac{d}{dx}(c)=0$$

   2. If f(x) is a differentiable function and c is a constant, then cf(x) is a differentiable function such that

      $$\frac{d}{dx}(cf(x))=c\frac{d}{dx}(f(x))$$

5. **Some useful results in finding derivatives**

1. $\sin 2x = 2\sin x\cos x$
2. $\cos 2x = 2\cos^2 x - 1$
3. $\cos 2x = 1 - 2\sin^2 x$
4. $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$
5. $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$
6. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$
7. $\sin 3x = 3\sin x - 4\sin^3 x$
8. $\cos 3x = 4\cos^3 x - 3\cos x$
9. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$
10. $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$
11. $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right\}$
12. $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left(\dfrac{x \pm y}{1 \mp xy}\right)$
13. $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, if $-1 \le x \le 1$
14. $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$, for all $x \in R$
15. $\sec^{-1} x + \operatorname{cosec}^{-1} x = \dfrac{\pi}{2}$, if $x \in (-\infty, -1] \cup [1, \infty)$
16. $\sin^{-1}(-x) = -\sin^{-1} x$, for $x \in [-1, 1]$
17. $\cos^{-1}(-x) = \pi - \cos^{-1} x$, for $x \in [-1, 1]$
18. $\tan^{-1}(-x) = -\tan^{-1} x$, for $x \in R$
19. $\sin^{-1} x = \operatorname{cosec}^{-1}\left(\dfrac{1}{x}\right)$ if $x \in (-\infty, -1] \cup [1, \infty)$
20. $\cos^{-1} x = \sec^{-1}\left(\dfrac{1}{x}\right)$ if $x \in (-\infty, -1] \cup [1, \infty)$
21. $\tan^{-1} x = \begin{cases} \cot^{-1}\left(\dfrac{1}{x}\right), & \text{if } x > 0 \\ -\pi + \cot^{-1}\left(\dfrac{1}{x}\right), & \text{if } x < 0 \end{cases}$
22. $\sin^{-1}(\sin\theta) = \theta$, if $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$
23. $\cos^{-1}(\cos\theta) = \theta$, if $0 \le \theta \le \pi$

24. $\tan^{-1}(\tan\theta) = \theta$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

25. $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \theta \neq 0$

26. $\sec^{-1}(\sec\theta) = \theta$, if $0 < \theta < \pi, \theta \neq \frac{\pi}{2}$

27. $\cot^{-1}(\cot\theta) = \theta$, if $0 < \theta < \pi$

6. **Substitutions useful in finding derivatives**

	If the expression is	then substitute
1.	$a^2 + x^2$	$x = a\tan\theta$ or $a\cot\theta$
2.	$a^2 - x^2$	$x = a\sin\theta$ or $a\cos\theta$
3.	$x^2 - a^2$	$x = a\sec\theta$ or $a\operatorname{cosec}\theta$
4.	$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
5.	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or, $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2\cos 2\theta$

MIND MAP : LEARNING MADE SIMPLE

CHAPTER - 5

Continuity and Differentiability

Continuous Function

Suppose f is a real function on a subset of the real numbers and let 'c' be a point in the domain of f.
Then f is continuous at c if $\lim_{x\to c} f(x) = f(c)$
A real function f is said to be continuous if it is continuous at every point in the domain of f. For eg: The function $f(x) = \frac{1}{x}, x \neq 0$ is continuous
Let 'C' be any non-zero real number, then $\lim_{x\to c} f(x) \lim_{x\to c} \frac{1}{x} = \frac{1}{c}$. For $c = 0, f(c) = \frac{1}{c}$ So $\lim_{x\to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g, f-g, f.g$ and $\frac{f}{g}$ are continuous at $x = c\,(g(c) \neq 0)$.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$
Every differentiable function is continuous, but the converse is not true.

Chain Rule

if $f = vou$, $t = u(x)$ and if both $\frac{dt}{dx}, \frac{dv}{dt}$ exists, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$.

Logarithmic differentiation

Let $y = f(x) = [u(x)]^{v(x)}$

$\log y = v(x) \log[u(x)]$

$\frac{1}{y} = v(x) \frac{1}{u(x)} u'(x) + v'(x) \log[u(x)]$

$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}\right] u'(x) + v'(x) \log[u(x)]$

For $e.g.$: Let $y = a^x$ Then $\log y = x \log a$

$\frac{1}{y} \cdot \frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a = a^x \log a.$

Some Standard derivatives

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ (ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ (iv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{1-x^2}}$ (vi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$

(vii) $\frac{d}{dx}(e^x) = e^x$ (viii) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Mean Value Theorem

if $f : [a,b] \to R$ continuous on $[a, b]$ and differentiable on (a, b).
Then $\exists$ some c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$
e.g. Let $f(x) = x^2$ defined in the interval [2, 4]. Since $f(x) = x^2$ is continuous in [2, 4] and differentiable in (2, 4) as $f'(x) = 2x$ defined in (2, 4). So,

$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{16 - 4}{4 - 2} = 6, c \in (2, 4).$

Rolle's theorem

if $f : [a,b] \to R$ is continuous on $[a, b]$ and differentiable on (a, b). Such that $f(a) = f(b)$, then $\exists$ some c in (a, b) s.t. $f'(c) = 0$.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x)$ *i.e.*,
$\frac{d^2y}{dx^2} = f''(x)$ is the second order derivative of y *w.r.t.x.*
For eg : if $y = 3x^2 + 2$, then $y' = 6x$ and $y'' = 6$.

Derivatives of functions in parametric form

Let $x = f(t)$, $y = g(t)$ be two functions of parameter 't'.

Then, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \left(\frac{dx}{dt} \neq 0\right)$

Jhus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ [provided $f'(t) \neq 0$]

For eg : if $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta.$

Important Questions

Multiple Choice questions-

1. The function

$$f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

is continuous at x = 0, then the value of 'k' is:

(a) 3

(b) 2

(c) 1

(d) 1.5.

2. The function f(x) = [x], where [x] denotes the greatest integer function, is continuous at:

(a) 4

(b)-2

(c) 1

(d) 1.5.

3. The value of 'k' which makes the function defined by

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0, \end{cases}$$

continuous at x = 0 is

(a) -8

(b) 1

(c) -1

(d) None of these.

4. Differential coefficient of sec ($\tan^{-1} x$) w.r.t. x is

(a) $\frac{x}{\sqrt{1+x^2}}$

(b) $\frac{x}{1+x^2}$

(c) x$\sqrt{1+x^2}$

(d) $\frac{1}{\sqrt{1+x^2}}$

5. If y = log $(\frac{1-x_2}{1+x_2})$ then $\frac{dy}{dx}$ is equal to:

(a) $\frac{4x^3}{1-x^4}$

(b) $\frac{-4x}{1-x^4}$

(c) $\frac{1}{4-x^4}$

(d) $\frac{-4x^3}{1-x^4}$

6.

If y = $\sqrt{sinx+y}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{cosx}{2y-1}$

(b) $\frac{cosx}{1-2y}$

(c) $\frac{sinx}{1-2y}$

(d) $\frac{sinx}{2y-1}$

7. If u = sin-1 $(\frac{2x}{1+x_2})$ and u = tan-1 $(\frac{2x}{1-x_2})$ then $\frac{dy}{dx}$ is

(a) 12

(b) x

(c) $\frac{1-x^2}{1+x^2}$

(d) 1

8. If x = t², y = t³, then $\frac{d^2y}{dx^2}$ is

(a) $\frac{3}{2}$

(b) $\frac{3}{4t}$

(c) $\frac{3}{2t}$

(d) $\frac{3t}{2}$

9. The value of 'c' in Rolle's Theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

(a) 1

(b) -1

(c) $\frac{3}{2}$

(d) $\frac{1}{3}$

10. The value of 'c' in Mean Value Theorem for the function $f(x) = x(x - 2)$, $x \in [1, 2]$ is

(a) $\frac{3}{2}$

(b) $\frac{2}{3}$

(c) $\frac{1}{2}$

(d) $\frac{3}{4}$

Very Short Questions:

1. If y = log (cos ex), then find $\frac{dy}{dx}$ (Delhi 2019)
2. Differentiate cos {sin $(x)_2$} w.r.t. x. (Outside Delhi 2019)
3. Differentiate $\sin^2(x^2)$ w.r.t. x^2. (C.B.S.E. Sample Paper 2018-19)
4. Find $\frac{dy}{dx}$, if y + siny = cos or.
5. If $y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right), -\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$ then find $\frac{dy}{dx}$.
6. Is it true that $x = e^{\log x}$ for all real x? (N.C.E.R.T.)
7. Differentiate the following w.r.t. x : 3^{x+2}. (N.C.E.R.T.)

8. Differentiate log (1 + θ) w.r.t. $\sin^{-1}\theta$.

9. If $y = x^x$, find $\frac{dy}{dx}$.

10.

If y = $\sqrt{2^x + \sqrt{2^x + \sqrt{2^x + \ldots\ldots + 0\infty}}}$ then prove that: $(2y - 1)\frac{dy}{dx} = 2^x \log 2$.

Short Questions:

1. Discuss the continuity of the function: f(x) = |x| at x = 0. (N.C.E.R.T.)

2. If f(x) = x + 1, find $\frac{d}{dx}$(fof)(x). (C.B.S.E. 2019)

3. Differentiate $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$ with respect to x. (C.B.S.E. 2018 C)

4. Differentiate: $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x. (C.B.S.E. 2018)

5. Write the integrating factor of the differential equation:

   $(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$. (C.B.S.E. 2019 (Outside Delhi))

6. Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left[\frac{5x+12\sqrt{1-x^2}}{13}\right]$ (A.I.C.B.S.E. 2016)

7. Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left[\frac{6x-4\sqrt{1-4x^2}}{5}\right]$ (A.I.C.B.S.E. 2016)

8. If $y = \{x + \sqrt{x^2 + a^2}\}^n$, prove that $\frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$

Long Questions:

1. Find the value of 'a' for which the function 'f' defined as:

$$f(x) = \begin{cases} a \sin\frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

   s continuous at x = 0 (CBSE 2011)

2. Find the values of 'p' and 'q' for which:

$$\begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ p, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x > \dfrac{\pi}{2}. \end{cases}$$

is continuous at x = 2 (CBSE 2016)

3. Find the value of 'k' for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

is continuous at x = 0 (A.I.C.B.S.E. 2013)

4. For what values of 'a' and 'b\ the function 'f' defined as:

$$f(x) = \begin{cases} 3ax + b & \text{if } x < 1 \\ 11 & \text{if } x = 1 \\ 5ax - 2b & \text{if } x > 1 \end{cases}$$

is continuous at x = 1. (CBSE 2011)

Assertion and Reason Questions-

1. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is true.
e) Both A and R are false.

Assertion(A): $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & x \ne 0 \\ 0 & x = 0 \end{cases}$ is continuous at x = 0.

Reason (R): Both h(x) = x^2, $g(x)=\begin{cases}\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0 & x=0\end{cases}$ are continuous at x = 0.

2. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is true.
e) Both A and R are false.

Assertion (A): The function $f(x)=\begin{cases}|x|+\sqrt{x-|x|}, & x\geq 0\\ \sin x & x<0\end{cases}$ is continuous everywhere.

Reason (R): f(x) is periodic function.

Case Study Questions-

1. If a relation between x and y is such that y cannot be expressed in terms of x, then y is called an implicit function of x. When a given relation expresses y as an implicit function of x and we want to find $\frac{dy}{dx}$, then we differentiate every term of the given relation w.r.t. x, remembering that a tenn in y is first differentiated w.r.t. y and then multiplied by $\frac{dy}{dx}$.

Based on the ab:ve information, find the value of $\frac{dy}{dx}$ in each of the following questions.

i. $x^3 + x^2y + xy^2 + y^3 = 81$

a. $\frac{(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$

b. $\frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$

c. $\frac{(3x^2+2xy-y^2)}{x^2-2xy+3y^2}$

d. $\frac{3x^2+xy+y^2}{x^2+xy+3y^2}$

ii. $x^y = e^{x-y}$

a. $\frac{x-y}{(1+\log x)}$

b. $\frac{x+y}{(1+\log x)}$

c. $\frac{x-y}{x(1+\log x)}$

d. $\frac{x+y}{x(1+\log x)}$

iii. $e^{\sin y} = xy$

a. $\frac{-y}{x(y\cos y-1)}$

b. $\frac{y}{y\cos y-1}$

c. $\frac{y}{y\cos y+1}$

d. $\frac{y}{x(y\cos y-1)}$

iv. $\sin^2 x + \cos^2 y = 1$

a. $\frac{\sin 2y}{\sin 2x}$

b. $-\frac{\sin 2x}{\sin 2y}$

c. $-\frac{\sin 2y}{\sin 2x}$

d. $\frac{\sin 2x}{\sin 2y}$

v. $y=(\sqrt{x})^{\sqrt{x}^{\sqrt{x}\ldots\infty}}$

a. $\frac{-y^2}{x(2-y\log x)}$

b. $\frac{y^2}{2+y\log x}$

c. $\frac{y^2}{x(2+y\log x)}$

d. $\frac{y^2}{x(2-y\log x)}$

1. If y = f(u) is a differentiable function of u and u = g(x) is a differentiable function of x, then y = f(g(x)] is a differentiable function of x and $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$. This rule is also known as CHAIN RULE.

Based on the above information, find the derivative of functions w.r.t. x in the following questions.

i. $\cos\sqrt{x}$

a. $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

b. $\frac{\sin\sqrt{x}}{2\sqrt{x}}$

c. $\sin\sqrt{x}$

d. $-\sin\sqrt{x}$

ii. $7^{x+\frac{1}{x}}$

a. $\left(\frac{x^2-1}{x^2}\right) \cdot 7^{x+\frac{1}{x}} \cdot \log 7$

b. $\left(\frac{x^2+1}{x^2}\right) \cdot 7^{x+\frac{1}{x}} \cdot \log 7$

c. $\left(\frac{x^2-1}{x^2}\right) \cdot 7^{x-\frac{1}{x}} \cdot \log 7$

d. $\left(\frac{x^2+1}{x^2}\right) \cdot 7^{x-\frac{1}{x}} \cdot \log 7$

iii. $\sqrt{\frac{1-\cos x}{1+\cos x}}$

a. $\frac{1}{2}\sec^2 \frac{x}{2}$

b. $-\frac{1}{2}\sec^2 \frac{x}{2}$

c. $\sec^2 \frac{x}{2}$

d. $-\sec^2 \frac{x}{2}$

iv. $\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right) + \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

a. $\frac{-1}{x^2+b^2} + \frac{1}{x^2+a^2}$

b. $\frac{1}{x^2+b^2} + \frac{1}{x^2+a^2}$

c. $\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}$

d. None of these.

v. $\sec^{-1} x + \operatorname{cosec}^{-1} \frac{x}{\sqrt{x^2-1}}$

a. $\frac{2}{\sqrt{x^2-1}}$

b. $\frac{-2}{\sqrt{x^2-1}}$

c. $\frac{1}{|x|\sqrt{x^2-1}}$

d. $\frac{2}{|x|\sqrt{x^2-1}}$

Answer Key-

Multiple Choice questions-

1. Answer: (b) 2
2. Answer: (d) 1.5.
3. Answer: (d) None of these.
4. Answer:

   (a) $\frac{x}{\sqrt{1+x^2}}$

5. Answer:

   (b) $\frac{-4x}{1-x^4}$

6. Answer:

   (a) $\frac{\cos x}{2y-1}$

7. Answer: (d) 1
8. Answer: (b) $\frac{3}{4t}$
9. Answer: (a) 1
10. Answer: (a) $\frac{3}{2}$

Very Short Answer:

1. Solution:

   We have: $y = \log(\cos e^x)$

   $\therefore \frac{dy}{dx} = \frac{1}{\cos e^x}(-\sin e^x) \cdot e^x$

   $= -e^x \tan e^x$

2. Solution:

   Let $y = \cos\{\sin(x)^2\}$.

   $\therefore \frac{dy}{dx} = -\sin\{\sin(x)^2\}. \frac{dy}{dx}\{\sin(x)^2\}$

   $= -\sin\{\sin(x)^2\}. \cos(x)^2 \frac{dy}{dx}(x^2)$

   $= -\sin\{\sin(x)^2\}. \cos(x)^2 2x$

   $= -2x \cos(x)^2 \sin\{\sin(x)^2\}$.

3. Solution:

   Let $y = \sin^2(x^2)$.

   $\therefore \frac{dy}{dx} = 2\sin(x^2)\cos(x^2) = \sin(2x^2)$.

4. Solution:

   We have: $y + \sin y = \cos x$.

   Differentiating w.r,t. x, we get:

   $\frac{dy}{dx} + \cos y. \frac{dy}{dx} = -\sin x$

   $(1 + \cos y)\frac{dy}{dx} = -\sin x$

   Hence, $\frac{dy}{dx} = -\frac{\sin x}{1+\cos y}$

   where $y \neq (2n + 1)\pi$, $n \in Z$.

5. Solution:

Here $y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right)$

Put 3x = sin θ.

$y = \sin^{-1}(2\sin\theta\cos\theta)$

$= \sin^{-1}(\sin 2\theta) = 2\theta$

$= 2\sin^{-1} 3x$

$$\frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}}$$

6. Solution:

The given equation is $x = e^{\log x}$

This is not true for non-positive real numbers.

[∵ Domain of log function is R+]

Now, let $y = e^{\log x}$

If y > 0, taking logs.,

$\log y = \log(e^{\log x}) = \log x.\log e$

$= \log x . 1 = \log x$

⇒ y = x.

Hence, $x = e^{\log x}$ is true only for positive values of x.

7. Solution:

Let $y = 3^{x+2}$.

$\frac{dy}{dx} = {}^{3x+2}.\log 3.\frac{d}{dx}(x+2)$

$= 3^{x+2}.\log 3.(1+0)$

$= 3^{x+2}.\log 3 = \log 3\,(3^{x+2})$.

8. Solution:

Let $y = \log(1+\theta)$ and $u = \sin^{-1}\theta$.

$$\therefore \frac{dy}{d\theta} = \frac{1}{1+\theta} \text{ and } \frac{du}{d\theta} = \frac{1}{\sqrt{1-\theta^2}}.$$

$$\therefore \quad \frac{dy}{du} = \frac{dy/d\theta}{du/d\theta}$$

$$= \frac{\frac{1}{1+\theta}}{\frac{1}{\sqrt{1-\theta^2}}} = \sqrt{\frac{1-\theta}{1+\theta}}$$

9. Solution:

Here y = x^x ...(1)

Taking logs., log y = log x^x

⇒ log y = x log x.

Differentiating w.r.t. x, we get:

$\frac{1}{y} \cdot \frac{dy}{dx}$ = x 1x + logx. (1)

= 1 + log x.

Hence, $\frac{dy}{dx}$ = y (1 + log x) dx

= x^x (1 + log x). [Using (1)]

10. Solution:

The given series can be written as:

$y = \sqrt{2^x + y}$

Squaring, $y^2 = 2^x + y$

⇒ $y^2 - y = 2^x$.

Diff. w.r.t. x, (2y -1) $\frac{dy}{dx}$ = 2^x log 2.

Short Answer:

1. Solution:

By definition, $f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0. \end{cases}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x)$$

$$= \lim_{h \to 0} (-(0-h))$$

$$= \lim_{h \to 0} (h) = 0.$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x)$$

$$= \lim_{h \to 0} (0+h)$$

$$= \lim_{h \to 0} (h) = 0.$$

Also $f(0) = 0.$

Thus $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0).$

[∵ *Each = 0*]

Hence 'f' is continuous at x = 0.

2. Solution:

We have : f(x) = x + 1 ...(1)

∴ fof(x) = f (f(x)) =f(x)+ 1

= (x + 1) + 1 = x + 2.

∴ $\frac{d}{dx}$(fof)(x).) = $\frac{d}{dx}$(x + 2) = 1 + 0 = 1.

3. Solution:

Let $y = \tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$

$$= \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right)$$

[*Dividing num. & denom. by cos x*]

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right) = \frac{\pi}{4} - x$$

Differentiating (1) w.r.t. x,

$$\Rightarrow \quad \frac{dy}{dx} = -1$$

4. Solution:

$$\text{Let } y = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right) = \frac{\pi}{2} - \frac{x}{2}.$$

$$\therefore \frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}.$$

5. Solution:

The given differential equation is:

$(\tan^{-1} y - x)\, dy = (1 + y^2)\, dx$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \text{ Linear Equation}$$

$$\therefore \text{I.F} = e^{\int \frac{1}{1+y^2} dx} = e^{\tan^{-1} y}$$

6. Solution:

$$\text{We have : } y = \sin^{-1}\left[\frac{5x+12\sqrt{1-x^2}}{13}\right]$$

$$= \sin^{-1}\left(\frac{5}{13}x + \frac{12}{13}\sqrt{1-x^2}\right)$$

$$= \sin^{-1}\left(x\sqrt{1-\left(\frac{12}{13}\right)^2} + \sqrt{1-x^2}\cdot\frac{12}{13}\right)$$

(*Note this step*)

$$= \sin^{-1} x + \sin^{-1}\frac{12}{13}$$

$[\therefore \sin^{-1}A + \sin^{-1}B = \sin^{-1}(A\sqrt{1-B^2} + B\sqrt{1-A^2})]$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + 0 = \frac{1}{\sqrt{1-x^2}}, |x| < 1.$$

7. Solution:

We have : $y = \sin^{-1}\left(\frac{6x - 4\sqrt{1-4x^2}}{5}\right)$

$$= \sin^{-1}\left(\frac{6x}{5} - \frac{4}{5}\sqrt{1-4x^2}\right)$$

$$= \sin^{-1}\left((2x).\frac{3}{5} - \frac{4}{5}\sqrt{1-(2x)^2}\right)$$

$$= \sin^{-1}\left((2x)\sqrt{1-\left(\frac{4}{5}\right)^2} - \left(\frac{4}{5}\right)\sqrt{1-(2x)^2}\right)$$

$$= \sin^{-1}(2x) - \sin^{-1}\frac{4}{5}.$$

Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{1-(4x^2)}}.(2) - 0 = \frac{2}{\sqrt{1-4x^2}}.$

8. Solution:

$y = \{x + \sqrt{x^2 + a^2}\}^n$ .... (1)

$$\therefore \frac{dy}{dx} = n\left\{x+\sqrt{x^2+a^2}\right\}^{n-1} \cdot \frac{d}{dx}\left\{x+\sqrt{x^2+a^2}\right\}$$

$$= n\left\{x+\sqrt{x^2+a^2}\right\}^{n-1} \cdot \left[1+\frac{1}{2\sqrt{x^2+a^2}}(2x+0)\right]$$

$$= n\left\{x+\sqrt{x^2+a^2}\right\}^{n-1}\left\{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right\}$$

$$= \frac{n\left\{x+\sqrt{x^2+a^2}\right\}^{n}}{\sqrt{x^2+a^2}} = \frac{ny}{\sqrt{x^2+a^2}}, \quad [\text{Using } (1)]$$

which is true.

Long Answer:

1. Solution:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} a\sin\frac{\pi}{2}(x+1)$$

$$= \lim_{h\to 0} a\sin\frac{\pi}{2}(0-h+1)$$

$$= a\sin\frac{\pi}{2}(0-0+1)$$

$$= a\sin\frac{\pi}{2} = a.1 = a$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\tan x - \sin x}{x^3}$$

$$= \lim_{h \to 0} \frac{\tan(0+h) - \sin(0+h)}{(0+h)^3}$$

$$= \lim_{h \to 0} \frac{\tan h - \sin h}{h^3}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \frac{1 - \cos h}{h^2} \cdot \frac{1}{\cos h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}}{h^2}$$

$$\cdot \lim_{h \to 0} \frac{1}{\cos h}$$

$$= 1 . \frac{1}{2} \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \cdot \frac{1}{\cos 0}$$

$$= 1 . \frac{1}{2} (1)^2 . \frac{1}{1} = \frac{1}{2}.$$

Also f(0) = a sin π/2 (0+1)

= a sin π/2 = a(1) = a

For continuity,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

⇒ a = 1/2 = a

Hence, a = ½

2. Solution:

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin^3 x}{3\cos^2 x}$$

$$= \lim_{h\to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)}$$

$$= \lim_{h\to 0} \frac{1-\cos^3 h}{3\sin^2 h}$$

$$= \lim_{h\to 0} \frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$$

$$= \lim_{h\to 0} \frac{1+\cos^2 h+\cos h}{3(1+\cos h)}$$

$$= \frac{1+1+1}{3(1+1)} = \frac{1}{2}.$$

$$\lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{x\to\frac{\pi}{2}^+} \frac{q(1-\sin x)}{(\pi-2x)^2}$$

$$= \lim_{h\to 0} \frac{q\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi-2\left(\frac{\pi}{2}+h\right)\right]^2}$$

$$= \lim_{h\to 0} \frac{q(1-\cos h)}{(\pi-\pi-2h)^2}$$

$$= \lim_{h\to 0} \frac{q(1-\cos h)}{4h^2}$$

$$= \lim_{h\to 0} \frac{q \cdot 2\sin^2\frac{h}{2}}{4h^2}$$

$$= \lim_{h\to 0} \frac{q}{8}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$$

$$= \frac{q}{8}(1)^2 = \frac{q}{8}.$$

Also f($\frac{\pi}{2}$,) = p

For continuity $\lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{x\to\frac{\pi}{2}^+} f(x)$

= f($\frac{\pi}{2}$,)

$\Rightarrow \frac{1}{2} = \frac{q}{8}$ = p

Hence p = 1/2 and q = 4

3. Solution:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}$$

$$= \lim_{x\to 0^-} \frac{(\sqrt{1+kx}-\sqrt{1-kx})(\sqrt{1+kx}+\sqrt{1-kx})}{x(\sqrt{1+kx}+\sqrt{1-kx})}$$

[*Rationalising Numerator*]

$$= \lim_{x\to 0^-} \frac{(1+kx)-(1-kx)}{x(\sqrt{1+kx}+\sqrt{1-kx})}$$

$$= \lim_{x\to 0^-} \frac{2kx}{x(\sqrt{1+kx}+\sqrt{1-kx})}$$

$$= \lim_{x\to 0^-} \frac{2k}{\sqrt{1+kx}+\sqrt{1-kx}} \qquad [\because x \neq 0]$$

$$= \frac{2k}{1+1} = k$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{2x+1}{x-1} = \lim_{h\to 0} \frac{2(0+h)+1}{(0+h)-1}$$

$$= \frac{2(0)+1}{0-1} = \frac{1}{-1} = -1$$

$$\text{Also } f(0) = \frac{2(0)+1}{0-1} = \frac{1}{-1} = -1.$$

For continuity $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$

$\Rightarrow$ k = -1 = -1

Hence k = -1

4. Solution:

$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} (3ax + b)$

$= \lim_{h\to0} (3a (1-h) + b]$

$= 3a(1 - 0) + b$

$= 3a + b$

$\lim_{x\to1^+} f(x) = \lim_{x\to1^+} (5ax - 2b)$

$= \lim_{h\to0} [5a (1+h) - 2b]$

$= 5a (1+0) - 2b$

$= 5a - 2b$

Also f(1) = 11

Since 'f' is continuous at x = 1,

$\therefore \lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1)$

$\Rightarrow 3a + b = 5a - 2b = 11.$

From first and third,

3a + b = 11 …………… (1)

From last two,

5a – 2b = 11 …………… (2)

Multiplying (1) by 2,

6a + 2b = 22 ………….. (3)

Adding (2) and (3),

11a = 33

$\Rightarrow$ a = 3.

Putting in (1),

3(3) + b = 11

⇒ b = 11 – 9 = 2.

Hence, a = 3 and b = 2.

Case Study Answers-

1. Answer :

i. (b) $\frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$

Solution:

$$x^3 + x^2y + xy^2 + y^3 = 81$$

$$\Rightarrow 3^2 + x^2\frac{dy}{dx} + 2xy + 2xy\frac{dy}{dx} + y^2 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow (x^2 + 2xy + 3y^2)\frac{dy}{dx} = -3x^2 - 2xy - y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(3x^2+2xy+y^2)}{x^2+2xy+3y^2}$$

ii. (c) $\frac{x-y}{x(1+\log x)}$

Solution:

$$x^y = e^{x-y} \Rightarrow y\log x = x - y$$

$$y \times \frac{1}{x} + \log x \cdot \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}[\log x + 1] = 1 - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x[1+\log x]}$$

iii. (d) $\frac{y}{x(y\cos y-1)}$

Solution:

$$e^{\sin y} = xy \Rightarrow \sin y = \log x + \log y$$

$$\Rightarrow \cos y\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left[\cos y - \frac{1}{y}\right] = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x(y\cos y-1)}$$

iv. (d) $\frac{\sin 2x}{\sin 2y}$

Solution:

$$\sin^2 x + \cos^2 y = 1$$

$$\Rightarrow 2\sin x\cos x + 2\cos y\left(-\sin y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin 2x}{-\sin 2y} = \frac{\sin 2x}{\sin 2y}$$

v. (d) $\frac{y^2}{x(2-y\log x)}$

Solution:

$$y = (\sqrt{x})^{\sqrt{x}^{\sqrt{x}}\ldots\infty} \Rightarrow y = (\sqrt{x})^y$$

$$\Rightarrow y = y(\log\sqrt{x}) \Rightarrow \log y = \frac{1}{2}(y\log x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[y\times\frac{1}{x} + \log x\left(\frac{dy}{dx}\right)\right]$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{1}{y} - \frac{1}{2}\log x\right\} = \frac{1}{2}\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x}\times\frac{2y}{(2-y\log x)} = \frac{y^2}{x(2-y\log x)}$$

2. Answer :

i. (a) $\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

Solution:

Let $y=\cos\sqrt{x}$

$\therefore \frac{dy}{dx}=\frac{d}{dx}(\cos\sqrt{x})=-\sin\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$

$=-\sin\sqrt{x}\times\frac{1}{2\sqrt{x}}=\frac{-\sin\sqrt{x}}{2\sqrt{x}}$

ii. (a) $\left(\frac{x^2-1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$

Solution:

Let $y=7^{x+\frac{1}{x}}\ \therefore \frac{dy}{dx}=\frac{d}{dx}\left(7^{x+\frac{1}{x}}\right)$

$=7^{x+\frac{1}{x}}\cdot\log 7\cdot\frac{d}{dx}\left(x+\frac{1}{x}\right)=7^{x+\frac{1}{x}}\cdot\log 7\cdot\left(1-\frac{1}{x^2}\right)$

$=\left(\frac{x^2-1}{x^2}\right)\cdot 7^{x+\frac{1}{x}}\cdot\log 7$

iii. (a) $\frac{1}{2}\sec^2\frac{x}{2}$

Solution:

Let $y=\sqrt{\frac{1-\cos x}{1+\cos x}}=\sqrt{\frac{1-1+2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}-1+1}}=\tan\left(\frac{x}{2}\right)$

$\therefore \frac{dy}{dx}=\sec^2\frac{x}{2}\cdot\frac{1}{2}=\frac{1}{2}\sec^2\frac{x}{2}$

iv. (b) $\frac{1}{x^2+b^2}+\frac{1}{x^2+a^2}$

Solution:

Let $y=\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

$\therefore \frac{dy}{dx}=\frac{1}{b}\times\frac{1}{1+\frac{x^2}{b^2}}\times\frac{1}{b}+\frac{1}{a}\times\frac{1}{1+\frac{x^2}{a^2}}\times\frac{1}{a}$

$=\frac{1}{x^2+b^2}+\frac{1}{x^2+a^2}$

v. (d) $\frac{2}{|x|\sqrt{x^2-1}}$

Solution:

Let $y=\sec^{-1}x+\operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}}$

Put $x=\sec\theta \Rightarrow \theta=\sec^{-1}x$

$\therefore y=\sec^{-1}(\sec\theta)+\operatorname{cosec}^{-1}\left(\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}\right)$

$=\theta+\sin^{-1}\left[\sqrt{1-\cos^2\theta}\right]$

$=\theta+\sin^{-1}(\sin\theta)=\theta+\theta=2\theta=2\sec^{-1}x$

$\therefore \frac{dy}{dx}=2\frac{d}{dx}(\sec^{-1}x)=2\times\frac{1}{|x|\sqrt{x^2-1}}$

$=\frac{2}{|x|\sqrt{x^2-1}}$